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Reduction Theorems for Systems of Ordinary Differential Equations with a Turning Point[1]

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INTRODUCTION

In the recent past several American and Japanese mathematicians, notably R. E. Langer [1], Wolfgang Wasow [2], K. Okubo [3], and Y. Sibuya [4], have developed and used an elegant and powerful formal algorithm for treating both nth order scalar and first order systems of differential equations with a turning point. Historically, Langer was the first to use the methods which we have alluded to above, and he has given a lucid description of this in [5].

In the present paper we extend a version of this algorithm given by W. Wasow [2] to a considerably wider class of first order systems with a turning point. The results so obtained include those of Okubo and Wasow as very special cases.

We will study systems of the form

$$\epsilon^h y' = A(z, \epsilon)\, y, \qquad \left(' = \frac{d}{dz}\right), \tag{1.1}$$

where h is a positive integer and $A(z, \epsilon)$ is an $n \times n$ matrix function of the two complex variables z and ϵ which in a region

$$D = \{(z, \epsilon) \mid |z| < \delta_0, 0 < |\epsilon| \leqslant \epsilon_0, |\arg \epsilon| \leqslant \theta_0\} \tag{1.2}$$

has a uniform asymptotic expansion of the form

$$A(z, \epsilon) \sim \sum_{r=0}^{\infty} \hat{A}_r(z)\, \epsilon^r, \qquad \text{as} \qquad \epsilon \to 0, (\epsilon \in D), \tag{1.3}$$

with the matrices $\hat{A}_r(z)$, $(r = 0, 1, 2, \ldots)$, holomorphic at $z = 0$. It is known that with no loss of generality we may assume that $\hat{A}_0(0)$ has only one

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eigenvalue of multiplicity n, and that with a trivial normalization we will have

$$\operatorname{tr}[A(z, \epsilon)] \equiv 0, \qquad (z, \epsilon) \in D, \qquad \operatorname{tr} = \text{trace}. \tag{1.4}$$

These nonrestrictive assumptions imply that $\hat{A}_0(0)$ is nilpotent.

By a linear transformation of the dependent variable we can further achieve, without restricting our generality, that $\hat{A}_0(0)$ is in Jordan canonical form with zeros down the main diagonal. (See [2] for a resumé of this.)

For the remainder of the paper we make the critical and restrictive hypothesis that $A(0, 0)$ has only one Jordan block. That is, we suppose

$$\hat{A}_0(0) = J = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & & \cdot & & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ \cdot & & & & & & 0 \\ & & & & & & 1 \\ 0 & \cdot & \cdot & \cdot & & & 0 \end{bmatrix}. \tag{1.5}$$

2. The Formal Algorithm

This algorithm can be presented as follows. Suppose we let $y = P(z, \epsilon)\, y^*$, where $P(z, \epsilon)$ is a matrix nonsingular for $z = \epsilon = 0$. (1.1) then becomes

$$\epsilon^h y^{*\prime} = P^{-1}(AP - \epsilon^h P')\, y^*. \tag{2.1}$$

Putting $P^{-1}(AP - \epsilon^h P') = B$, we then wish to solve the equation

$$\epsilon^h P' = AP - PB. \tag{2.2}$$

The matrix $P(z, \epsilon)$ is to have the same regularity as the matrix $A(z, \epsilon)$ of (1.1) and is to be nonsingular at $z = \epsilon = 0$, while $B(z, \epsilon)$ is to be as simple as is possible.

One inserts formal power series for the matrices B and P, and the asymptotic series for A into (2.2). But here a variation occurs. Let us define a new parameter $\beta = \epsilon^h$. Then (2.2) has the form

$$\beta P'(z, \epsilon, \beta) = A(z, \epsilon)\, P(z, \epsilon, \beta) - P(z, \epsilon, \beta)\, B(z, \epsilon, \beta).$$

Suppose that we formally put

$$B(z, \epsilon, \beta) = \sum_{r=0}^{\infty} B_r(z, \epsilon)\, \beta^r,$$

and

$$P(z, \epsilon, \beta) = \sum_{r=0}^{\infty} P_r(z, \epsilon)\, \beta^r.$$

If we perform the indicated differentiations and multiplications followed by collection of like powers of β, we are led to the following recursion formulas where we have put $B_0(z, \epsilon) = A_0(z, \epsilon)$.

$$A_0(z, \epsilon)\, P_0(z, \epsilon) - P_0(z, \epsilon)\, A_0(z, \epsilon) = 0 \tag{2.3}$$

$$A_0(z, \epsilon)\, P_1(z, \epsilon) - P_1(z, \epsilon)\, A_0(z, \epsilon) = P_0' - [A_1(z, \epsilon)\, P_0 - P_0 B_1(z, \epsilon)]. \tag{2.4}$$

$$\begin{aligned} &A_0(z, \epsilon)\, P_\mu - P_\mu A_0(z, \epsilon) \\ &= P_{\mu-1} - [A_1 P_{\mu-1} - P_{\mu-1} B_1] + P_0 B_\mu - F_\mu(z, \epsilon), \qquad \mu > 1. \end{aligned} \tag{2.5}$$

Here

$$A_r(z, \epsilon) = \sum_{j=0}^{h-1} \hat{A}_{j+r}(z)\, \epsilon^j, \qquad (r = 0, 1, 2, \ldots), \tag{2.6}$$

while

$$\begin{aligned} F_\mu(z, \epsilon) &= A_\mu P_0 & (\mu = 2), \\ &= \sum_{j=2}^{\mu-1} (A_j P_{\mu-j} - P_{\mu-j} B_j) + A_\mu P_0 & (\mu > 2). \end{aligned} \tag{2.7}$$

The central idea is to obtain, by a proper choice of the matrices $B_j(z, \epsilon)$, $(j = 1, 2, \ldots)$, formal solutions for the Eq. (2.3) through (2.5) of the form

$$P_\mu(z, \epsilon) = \sum_{r=0}^{\infty} P_{\mu r}(z)\, \epsilon^r, \qquad (\mu = 0, 1, \ldots),$$

where the matrices $P_{\mu r}(z)$ are to be holomorphic for $|z| < \delta_0$. The matrix $P_{00}(z)$ is to be nonsingular for $|z| < \delta_0$. The matrices $B_j(z, \epsilon)$, $(j = 1, 2, 3, \ldots)$ are to be as simple as possible; polynomials in z, for example.

Equation (2.4) is a singular linear system of algebraic equations since the homogeneous equation $A_0(z, \epsilon)\, P_1 - P_1 A_0(z, \epsilon) = 0$ always has nontrivial solutions.

W. Wasow in [2] used properties of the linear operator

$$L_{A_0} X = A_0(z, \epsilon)\, X - X A_0(z, \epsilon) \tag{2.8}$$

to satisfy the necessary and sufficient conditions of compatibility for the right hand sides of (2.4) and (2.5).

Before we explore this matter in greater detail, let us first note the following Lemma.

LEMMA 2.1. *Suppose $C(z, \epsilon)$ is an $n \times n$ matrix of the form*

$$C(z, \epsilon) = \sum_{j=0}^{\infty} C_j(z)\, \epsilon^r, \qquad (| \epsilon | < \epsilon_0),$$

where the matrices $C_j(z)$ are holomorphic for $| z | < \delta_0$. If $C(0, 0) = J$, (see (1.5)) then there exist positive constants ϵ_1 and δ_1 and a nonsingular matrix

$$Q(z, \epsilon) = \sum_{j=0}^{\infty} Q_j(z)\, \epsilon^j, \qquad (| \epsilon | < \epsilon_1),$$

where the matrices $Q_j(z)$, $(j = 0, 1, \ldots)$, are all holomorphic for $| z | < \delta_1$, such that $Q(z, \epsilon)^{-1} C(z, \epsilon) Q(z, \epsilon)$ is the companion matrix formed with the characteristic polynomial of $C(z, \epsilon)$. (See [6], p. 149.)

The major difficulty which arises in the proof of Lemma 2.1 is that while it is perfectly plausible for $C(z, \epsilon)$ to be similar, for each fixed point (z, ϵ), to the companion matrix formed with the characteristic polynomial of $C(z, \epsilon)$, there is no guarantee of the existence of a matrix $Q(z, \epsilon)$, nonsingular at $z = \epsilon = 0$, with the same regularity as $C(z, \epsilon)$, which performs the similarity transformation throughout the region $| z | < \delta_1$ and $| \epsilon | < \epsilon_1$.

In [7] W. Wasow proved that if the null-space of the operator L_{A_0} is of constant dimension in a neighborhood of $z = \epsilon = 0$, then pointwise similarity implies *analytic similarity* in the sense specified in the statement of the lemma. Wasow's proof in [7] is actually given only for a function of one complex variable, but the more general statement is easily verified by only making a slight change of notation in his proof.

Let us suppose more generally that the matrix A_0 is only *continuous* at $z = \epsilon = 0$. We can prove that if the null-space of L_{A_0} is of dimension n at $z = \epsilon = 0$, then it is of dimension n in a full neighborhood of $z = \epsilon = 0$. To this end let $n_i = n_i(z, \epsilon)$, $(i = 1, \ldots, n)$, be the degrees of the invariant polynomials of the matrix $A_0(z, \epsilon)$ arranged in decreasing order.

For each point $(z, \epsilon) \in D$, the dimension of the null-space of L_{A_0} is

$$N = N(z, \epsilon) = \sum_{k=1}^{n} (2k - 1)\, n_k .$$

(See [6], p. 222).

At $z = \epsilon = 0$, (1.5) implies $N = n_1 = n$, so that the *rank* of the operator L_{A_0} is $n^2 - n$ at $z = \epsilon = 0$. Hence some minor of order $n^2 - n$ of a matrix representation for L_{A_0} does not vanish in a full neighborhood of $z = \epsilon = 0$, so that $\text{Rank}\,(L_{A_0}) \geqslant n^2 - n$ near $z = \epsilon = 0$.

Since the product of the invariant polynomials is the characteristic polynomial, we see that $n \leqslant N(z, \epsilon)$ for all $(z, \epsilon) \in D$.

From the rank-nullity theorem

$$\begin{aligned} n^2 &= \text{Rank}\,(L_{A_0}) + N \\ &\geqslant n^2 + N - n \\ &\geqslant n^2 \end{aligned}$$

near enough to $z = \epsilon = 0$. It follows from the above equality that $N(z, \epsilon) = n$. From this, and the fact that the characteristic polynomial is the product of the invariant polynomials, we conclude that

$$2n_2 + 4n_3 + \cdots + (2n - 2)\, n_n = 0$$

near enough to $z = \epsilon = 0$.

Since the integers n_j, $(j = 2, \ldots, n)$, are nonnegative they must be zero; hence we also have $n_1 = n$ near enough to $z = \epsilon = 0$. Thus the companion matrix formed with the characteristic polynomial of $C(z, \epsilon)$ and $C(z, \epsilon)$ itself are pointwise similar near enough to $z = \epsilon = 0$.

Wasow's theorem in [7] can now be used to complete the proof of Lemma 2.1.

Hence, with a certain nonsingular transformation $y = Q(z, \epsilon)\, y^*$,

$$Q(z, \epsilon) = \sum_{j=0}^{\infty} Q_j(z)\, \epsilon^j,$$

we may assume, with no loss of generality, that $A_0(z, \epsilon)$ is now a companion matrix. Notice that the characteristic polynomial for $A_0(z, \epsilon)$ remains invariant under changes of the dependent variable for all values of the positive integer h.

We introduce the inner product

$$(X, Y) = \text{tr}\,(XY^*) \tag{2.9}$$

on the space of $n \times n$ matrices with complex entries. Here, as usual, Y^* denotes the conjugate transpose of the matrix Y.

The adjoint to the linear operator L_{A_0}, which we designate $L_{A_0}^*$, is then given by

$$L_{A_0}^* X = A_0{}^* X - X A_0{}^* = L_{A_0^*} X, \tag{2.10}$$

as may be easily verified.

We note that the null-space of $L^*_{A_0}$ is of dimension n in a sufficiently small neighborhood of $z = \epsilon = 0$.

This yields

LEMMA 2.2. *For z and ϵ near enough to zero, the matrices* $(A_0{}^*(z, \epsilon))^j$, $(j = 0,..., n-1)$, *span the null-space of* $L^*_{A_0}$.

PROOF. The null-space of $L^*_{A_0}$ is of constant dimension n near enough to $z = \epsilon = 0$.

The n linearly independent matrices $(A_0{}^*(0, 0))^j = (J^T)^j$, $(j = 0,..., n-1)$, are in the null-space of $L^*_{A_0}$ for $z = \epsilon = 0$. By a continuity argument the lemma follows.

LEMMA 2.3. *Suppose $K(z, \epsilon)$ is an $n \times n$ matrix function holomorphic in z and ϵ in the region (1.2) and there possesses the same type of asymptotic expansion as the matrix $A(z, \epsilon)$ as $\epsilon \to 0$, $(\epsilon \in D)$. If $L_{A_0}X = K(z, \epsilon)$ has particular solutions for $X(z, \epsilon)$ for every $|z| < \delta_0$, $|\epsilon| < \epsilon_0$, then it has a particular solution with the same regularity as $K(z, \epsilon)$.*

Lemma 2.3 is a mild generalization of Lemma 3.3 of [2] in the sense that it really is a property of a subalgebra of functions which locally contains its reciprocals.

3. CLASSIFICATION OF SOME PROBLEM TYPES AND OTHER PRELIMINARIES

Let

$$\det (A_0(z, 0) - \lambda I) = (-1)^n (\lambda^n - \lambda^{n-1} a_1{}^*(z) - \cdots - \lambda a^*_{n-1}(z) - a_n{}^*(z)). \tag{3.1}$$

By (1.5), for some positive integer k_j,

$$a_j{}^*(z) = z^{k_j} a_j(z), \tag{3.2}$$

$(j = 1,..., n)$, with $a_1(z) \equiv 0$ by (1.4). The functions $a_j(z)$, $(j = 1,..., n)$, are all holomorphic at $z = 0$.

We may further suppose that by choosing the integers k_j sufficiently large, at least one of the holomorphic functions $a_j(z)$ is different from zero at $z = 0$. (If not, all the functions $a_j{}^*$ vanish identically. By application of certain "shearing" transformations, $y = \operatorname{diag}(1, \epsilon^g,..., \epsilon^{(n-1)g})\, y^*$, with g chosen carefully, the system (1.1) splits into two or more systems of lower order, the pole with respect to the parameter ϵ may be decreased, or the system can be rewritten in a manner equivalent to (1.1), where one of the functions $a_j{}^*$ does not vanish identically. See [8] for a precise statement and proof of this.)

To ease the notation, let the turning point problems satisfying (1.5) be denoted by the symbol $T_n(h, k, \nu)$.

Here n denotes the dimension of the system, h is the order of the pole with respect to the parameter ϵ, while

$$k = \min_j k_j \qquad \text{and} \qquad \nu = \max \{ j \mid k_j = k \}.$$

In terms of the polynomial (3.1), k is obtained by finding the functions $\{a_j{}^*(z)\}$ with the smallest order of zero at $z = 0$. The integer ν is then obtained by finding the function $a_\nu{}^*$ which is right-most in the right member of (3.1), and which has a zero of exact order k at $z = 0$.

We confine our attention in this paper to the two important special cases where

Case 1:

$$\nu = n.$$

Case 2:

$$\nu = n - 1.$$

Notice that these two cases cover all second and third order systems satisfying (1.5).

In our computations throughout this paper, we will always assume $h = 1$. By using the methods of this paper, it is possible to obtain a reduction of (1.1) satisfying (1.5) for values of the integer $h > 1$. This will not be done here, but the techniques used are quite similar.

However, for $h = 1$, the recursion formulas (2.3) through (2.5) no longer depend on ϵ, and hence are simpler to satisfy. Also, the introduction of a new parameter $\beta = \epsilon$ is no longer necessary.

4. Lemmas for Computation of the Reduced Equation

In this section we will prove several lemmas which enable us to formally obtain a simplified version of the differential equation (1.1).

Lemma 4.1. *Put $A_0{}^r(z, 0) = J^r + E_r(z)$, where J is the Jordan matrix of (1.5). Then $E_r(z)$ is identically zero above and on the* rth *parallel to the main diagonal,* $(r = 1,..., n - 1)$.

PROOF. This is directly verified for $r = 1$ and $r = 2$. Assume the lemma true for all r up to $r = m - 2 < n - 1$. Then

$$\begin{aligned} A_0^{m-1} &= A_0 A_0^{m-2} \\ &= [J + E_1]\,[J^{m-2} + E_{m-2}] \\ &= J^{m-1} + E_1 J^{m-2} + J E_{m-2} + E_1 E_{m-2} \\ &= J^{m-1} + E_{m-1}\,. \end{aligned}$$

Applying the induction hypothesis, JE_{m-2} has zeros on (and above) the $(m-1)$st super diagonal. Noting that both $E_1 E_{m-2}$ and $E_1 J^{m-2}$ have nonzero entries only in the last row, the lemma is proved.

Notice that if the coefficient functions in the characteristic polynomial for $A_0(z, 0)$ all have zeros at $z = 0$ of order $k \geqslant 1$, we have $E_r(z) = 0(z^k)$, $(r = 1,\dots, n - 1)$.

LEMMA 4.2. *Suppose all the coefficients in the characteristic polynomial for $A_0(z)$ are divisible by z^k.*

Let $B(z) = \{b_{ij}(z)\}$ be any matrix holomorphic at $z = 0$ with nonzero entries only in the last row.

Then $\operatorname{tr}(BA_0^r(z)) = b_{n,n-r}(z) + g_r(z)$, $(r = 0,\dots, n - 1)$, *where* $g_r(z)$ *involves only* $b_{n,n}(z),\dots, b_{n,n-r+1}(z)$, *from the last row of* $B(z)$, *and* $g_r(z) = 0(z^k)$, $(r = 0,\dots, n - 1)$.

If $r \geqslant n$, then

$$\operatorname{tr}(BA_0^r(z)) = \sum_{j=1}^{n} f_{jr}(z)\, b_{nj}(z) + 0(z^{2k}),$$

where the holomorphic functions f_{jr} are $0(z^k)$.

PROOF. It is only necessary to prove the first part of the lemma, since the second part follows from the first by successive applications of the Cayley-Hamilton theorem.

By assumption,

$$B(z) = \begin{bmatrix} & 0 & \\ b_{n1}\,, & \cdots, & b_{nn} \end{bmatrix}.$$

The lemma is true for $r = 0$ and $r = 1$. Assume it true for all r up to $r = m - 2 < n - 1$.

Then by putting

$$B = B_1 + B_2\,,$$

where

$$B_1 = \begin{bmatrix} & & 0 & & & \\ 0, & \cdots, & b_{n-m+1}\,, & 0, & \cdots, & 0 \end{bmatrix}$$

and

$$B_2 = B - B_1,$$

we have

$$\operatorname{tr}(BA_0^{m-1}) = \operatorname{tr}(B_1A_0^{m-1}) + \operatorname{tr}(B_2A_0^{m-1}).$$

Now from Lemma 4.1,

$$B_1A_0^{m-1} = B_1J^{m-1} + B_1E_{m-1},$$

where

$$\operatorname{tr}(B_1E_{m-1}) = 0.$$

Since

$$\operatorname{tr}(B_1J^{m-1}) = b_{n-m+1},$$

setting

$$g_{m-1}(z) = \operatorname{tr}(B_2A_0^{m-1}(z)) = \operatorname{tr}(B_2E_{m-1}(z)) = 0(z^k)$$

completes the proof of the lemma.

LEMMA 4.3. *Suppose that all coefficients* $\{a_j{}^*(z)\}$, *of the characteristic polynomial of* $A_0(z, 0)$ *are divisible by* z^k, $(k \geqslant 1)$. *Then*

$$\operatorname{tr}(A_0{}^j) = \begin{cases} n, & j = 0 \\ ja_j{}^*(z) + 0(z^{2k}), & (0 < j \leqslant n). \end{cases}$$

PROOF. This is certainly true for $j = 0$. It is directly verified to be true for $j = 1$ and $j = 2$. Assume the lemma true for all j up to $m - 1$. Then by Newton's formulas and the Cayley-Hamilton theorem (see [6], p. 87),

$$\begin{aligned} \operatorname{tr} A_0{}^m &= a_2{}^* \operatorname{tr} A_0^{m-2} + \cdots + a_{m-2}^* \operatorname{tr} A_0{}^2 + ma_m{}^* \\ &= 0(z^k)\,[\operatorname{tr} A_0^{m-2} + \cdots + \operatorname{tr} A_0{}^2] + ma_m{}^* \\ &= ma_m{}^*(z) + 0(z^{2k}). \end{aligned}$$

Here the second equality uses the assumption that all coefficients are divisible by z^k, while the last equality is a result of the induction hypothesis. This completes the proof of the lemma.

LEMMA 4.4. *Suppose the hypotheses of Lemma 4.3 are satisfied, and suppose* $p > 0$. *Then*

$$\operatorname{tr}(A_0^{n+p}) = 0(z^{2k}).$$

PROOF. This follows for $p = 1$ by an application of the Cayley-Hamilton theorem:

$$A_0^{n+1} = a_1{}^*A_0{}^n + \cdots + a_n{}^*A_0,$$

and Lemma 4.3. The general statement is a consequence of an obvious induction proof which we omit here.

LEMMA 4.5. *Suppose the hypotheses of Lemma 4.3 are satisfied. Suppose* $a_\nu{}^*(z) = 0(z^{k+1})$ *for* $\nu = n,\dots, j+1$. *Then if* $n < 2j < 2n$,

$$\operatorname{tr} A_0^{2j} = j[a_j{}^*(z)]^2 + 0(z^{2k+1}).$$

PROOF. Put $2j = n + p$, where $p > 0$. Then the Cayley-Hamilton theorem yields

$$\begin{aligned} A_0^{2j} = A_0^{n+p} &= a_1{}^* A_0^{n+p-1} + \cdots + a_j{}^* A_0^{n-j+p} + \cdots + a_n{}^* A_0{}^p \\ &= a_1{}^* A_0^{n+p-1} + \cdots + a_{j-1}^* A_0^{j+1} + a_j{}^* A_0{}^j + \cdots + a_n{}^* A_0{}^p. \end{aligned}$$

By hypothesis and Lemmas 4.3 and 4.4, $\operatorname{tr}(A_0^{j+m}) = 0(z^{k+1})$, $(m > 0)$. Hence

$$a_1{}^* \operatorname{tr} A_0^{n+p-1} + \cdots + a_{j-1}^* \operatorname{tr} A_0^{j+1} = 0(z^{2k+1}).$$

Again by the hypothesis and Lemma 4.3

$$a_{j+1}^* \operatorname{tr} A_0^{j-1} + \cdots + a_n{}^* \operatorname{tr} A_0{}^p = 0(z^{k+1}) [\operatorname{tr} A_0^{j-1} + \cdots + \operatorname{tr} A_0{}^p] = 0(z^{2k+1}).$$

Finally, since $j < n$,

$$\operatorname{tr} A_0{}^j = j a_j{}^*(z) + 0(z^{2k}),$$

so that

$$a_j{}^* \operatorname{tr} A_0{}^j = j[a_j{}^*(z)]^2 + 0(z^{3k}).$$

Thus

$$\begin{aligned} \operatorname{tr} A_0^{2j} &= j[a_j{}^*(z)]^2 + 0(z^{2k+1}) + 0(z^{3k}) \\ &= j[a_j{}^*(z)]^2 + 0(z^{2k+1}). \end{aligned}$$

5. REDUCTION THEOREMS FOR TURNING POINT PROBLEMS OF TYPE $T_n(1, k, n)$

In this section we will prove

THEOREM 5.1. *Every turning point problem of type* $T_n(1, k, n)$ *can be formally reduced by a nonsingular transformation* $y = P(z, \epsilon)\, y^*$ *to the form* $\epsilon y^{*\prime} = B(z, \epsilon)\, y^*$, *where*

$$B(z, \epsilon) = A_0(z) + \epsilon B_1(z, \epsilon),$$

with

$$B_1(z, \epsilon) = \sum_{j=0}^{k-2} \begin{bmatrix} 0 & 0 & \cdots & & 0 & 0 & 0 \\ \vdots & & \cdot & & & & \vdots \\ 0 & & & \cdot & & & \\ \vdots & & & & \cdot & & \\ b_{1j}(\epsilon), & & \cdots, & & b_{n-1,j}(\epsilon), & & 0 \end{bmatrix} z^j.$$

The $b_{mj}(\epsilon)$, $(j = 0,..., k-2)$, $(m = 1,..., n-1)$, *are formal power series in* ϵ *with constant coefficients.*

$P(z, \epsilon)$ *is a formal power series in* ϵ *with coefficients which are holomorphic for* $|z| < \delta_0$. $P(z, 0)$ *is nonsingular for* $|z| < \delta_0$.

Wolfgang Wasow proved Theorem 5.1 in [2] for problems of type $T_n(1, 1, n)$, and we use a modified version of his methods in the present paper.

We now prove Theorem 5.1.

Equation (2.3) (for $h = 1$, when ϵ does not occur) may be solved by putting

$$P_0(z) = \sum_{i=1}^{n} q_i(z)\, A_0^{n-i}(z). \tag{5.1}$$

Here the scalar functions q_i, $(i = 1,..., n)$, are arbitrary. It is implicit in the proof of Lemma 2.1 that (5.1) is the most general solution of (2.3). As we noted in Section 2, Eq. (2.4) is a singular system of algebraic equations. *Pointwise, then,* the right-hand side of (2.4) must be orthogonal to the null space of the operator adjoint to L_{A_0} in order that this equation have solutions.

By using Lemma 2.2, these necessary and sufficient conditions of compatibility are seen to be

$$(P_0' - (A_1P_0 - P_0B_1), (A_0{}^*)^j) = 0, \qquad (j = 0,..., n-1). \tag{5.2}$$

(5.2) is seen to be the following system of ordinary differential equations for the scalar functions q_i, $(i = 1,..., n)$:

$$S(z)q' = T(z)\, q, \tag{5.3}$$

where

$$q = (q_1, ..., q_n)^T$$

and

$$S(z) = \{\operatorname{tr} A_0^{n-k+r}\}, \qquad \text{columns:} \quad (k = 1,..., n),$$
$$\text{rows:} \qquad (r = 1, 2,..., n-1, 0), \tag{5.4}$$

while

$$T(z) = \{\mathrm{tr}\,[A_0^{n-k+r}(A_1 - B_1)] - (n-k)\,\mathrm{tr}\,[A_0^{n-k+r-1}\,A_0{}']\} \qquad (k = 1,\dots, n),$$

$$(r = 1, 2,\dots, n-1, 0),$$

$$= T_1(z) - T_2(z). \tag{5.5}$$

Notice that the rows of the matrix $S(z)$ of (5.4) are not quite naturally ordered; the first row being written last.

The definitions of the matrices T_1 and T_2 are self-explanatory.

From Lemmas 4.3 and 4.4 we find that

$$S(z) = \begin{bmatrix} nz^k a_n, & & \cdots & \\ & nz^k a_n, & & 0(z^k) \\ & & \cdot & \\ & 0 & \cdot & \\ & & & \cdot \\ & & & nz^k a_n \\ (n-1)\,z^k a_{n-1}, & \cdots, & a_1, & n \end{bmatrix} + 0(z^{2k}). \tag{5.6}$$

Hence, $\mathrm{diag}\,(1, 1,\dots, 1, z^k a_n(z))\, S(z) = z^k M_1(z)$, where

$$M_1(z) = \begin{bmatrix} na_n & & & \\ & \cdot & & 0(1) \\ & & \cdot & \\ & & & \cdot \\ 0 & & & na_n \\ & & & & na_n \end{bmatrix} + 0(z^k). \tag{5.7}$$

Note that $M_1(z)$ is invertible if z is near enough to $z = 0$, since $a_n(0) \neq 0$. Put

$$a_{1j} = \mathrm{tr}\,(A_1 A_0^{n-j}), \qquad (j = 1,\dots, n).$$

Now by Lemma 4.2

$$T_1(z) = \begin{bmatrix} 0 & a_{11} - b_{11}, & \cdots, & a_{1n-1} - b_{1n-1} \\ 0 & 0 & & \\ \cdot & \cdot & & \\ \cdot & & \cdot & \\ \cdot & & & \cdot \\ 0 & & 0, & a_{11} - b_{11} \\ *, & *, \quad \cdots, & *, & a_{1n} - b_{1n} \end{bmatrix} + 0(z^k), \tag{5.8}$$

and

$$T_2(z) = z^{k-1} \begin{bmatrix} k(n-1)\,a_n & & & & \\ 0 & k(n-2)\,a_n & & 0(1) & \\ 0 & & \cdot & & \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ \cdot & & & ka_n & \\ 0, & & \cdots, & 0 & 0 \end{bmatrix} + 0(z^k). \quad (5.9)$$

Let $\sum_{r=0}^{\infty} a_{1jr} z^r$, $(j = 1,\ldots, n)$, be the power series expansions of the functions $a_{1j}(z)$ about $z = 0$.

If we put $b_{1n}(z) = 0$ and

$$b_{1j}(z) = \sum_{r=0}^{k-2} a_{1jr} z^r, \qquad (j = 1,\ldots, n - 1), \tag{5.10}$$

then the system (5.3) takes the form

$$z^k M_1(z)\, q' = \operatorname{diag}(1, 1,\ldots, 1, z^k a_n(z))\; T(z)\, q = z^{k-1} R(z)\, q, \tag{5.11}$$

where

$$R(z) = - \begin{bmatrix} k(n-1)\,a_n & & & & \\ 0 & k(n-2)\,a_n & & 0(1) & \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ \cdot & & & & \cdot \\ & & & ka_n & \\ 0, & 0, & \cdots, & 0, & 0 \end{bmatrix} + 0(z) \tag{5.12}$$

is a matrix with holomorphic entries.

Hence, (5.11) becomes

$$\begin{aligned} zq' &= M_1^{-1}(z)\, R(z)\, q \\ &= H(z)\, q, \end{aligned} \tag{5.13}$$

where

$$H(z) = \begin{bmatrix} -\left(\frac{n-1}{n}\right) k & & & & \\ 0 & -\left(\frac{n-2}{n}\right) k & & * & \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ \cdot & & & & \cdot \\ & & & \frac{-k}{n} & \\ 0, & \cdots, & 0, & 0, & 0 \end{bmatrix} + 0(z) \tag{5.14}$$

is a matrix with entries holomorphic at $z = 0$.

Since no eigenvalue of $H(0)$ is a positive integer while one eigenvalue is zero, there exists a formal power series solution for (5.13) where we take $q_n(0) = 1$. By a classical theorem from the theory of differential equations with a regular singular point, this formal series converges for $|z| < \delta_0$. Also, with the above mentioned choice for $q_n(0)$, the matrix $P_0(z)$ of (5.1) is nonsingular for $|z| < \delta_0$, if δ_0 is small enough. This is seen by noting that

$$\begin{aligned} P_0(0) &= q_n(0)\, I + K \\ &= I + K, \end{aligned} \tag{5.15}$$

where K is strictly upper triangular.

For future reference, let $\tilde{q}_n(z)$ denote the last component of the vector $q(z)$.

In the preceding paragraphs we determined $P_0(z)$ in such a way that Eq. (2.4) has a particular solution for each z with $|z| < \delta_0$. By Lemma 2.3, we may find a particular holomorphic solution, say $\tilde{P}_1(z)$, for (2.4). To this particular solution we may add an arbitrary solution of the homogeneous equation $L_{A_0}P = 0$.

From Lemma 2.1, the most general solution of (2.4) is then

$$P_1(z) = \tilde{P}_1(z) + \mathring{P}_1(z), \tag{5.16}$$

where

$$\mathring{P}_1(z) = \sum_{k=1}^{n} q_k(z)\, A_0^{n-k}(z). \tag{5.17}$$

The scalar functions $q_k(z)$, $(k = 1,\ldots, n)$, are arbitrary, and are, in general, not the same as those in (5.1).

Having solved Eq. (2.4) we will now solve Eq. (2.5) with $\mu = 2$.

The right-hand side of (2.5) with $\mu = 2$ must satisfy the conditions

$$\operatorname{tr}\,[(P_1' - (A_1P_1 - P_1B_1) + P_0B_2 - F_2)\, A_0{}^k] = 0, \quad (k = 1, 2,\ldots, n - 1, 0). \tag{5.18}$$

(5.18) is seen to be the following system of differential equations for q_k, $(k = 1,\ldots, n)$.

$$S(z)\, q' = T(z)\, q + \phi_2(z), \tag{5.19}$$

where

$$\phi_2(z) = \{-\operatorname{tr}\,(P_0B_2 - \hat{F}_2]\, A_0{}^k)\}, \qquad (k = 1,\ldots, n - 1, 0), \tag{5.20}$$

with

$$\hat{F}_2 = F_2 + \tilde{P}_1' - (A_1\tilde{P}_1 - \tilde{P}_1B_1).$$

Let $\overset{\circ}{f}_{2j}$, $(j = 1,\dots, n)$, denote the functions $\operatorname{tr}(\hat{F}_2 A_0^{n-j})/\tilde{q}_n(z)$. Expanding these functions in power series,

$$\overset{\circ}{f}_{2j}(z) = \sum_{\nu=0}^{\infty} f_{2j\nu} z^{\nu}, \qquad (j = 1,\dots, n), \tag{5.21}$$

let us define the matrix $B_2(z)$ to be as follows:

Make every entry zero except possibly in the last row. In the last row set

$$b_{2j}(z) = \sum_{\nu=0}^{k-2} \overset{\circ}{f}_{2j\nu} z^{\nu}, \qquad (\text{for } j = 1,\dots, n-1) \tag{5.22}$$

and put $b_{2n}(z) = 0$. Now multiply the system (5.19) through by the diagonal matrix

$$\operatorname{diag}(1, 1,\dots, 1, z^k a_n(z)).$$

We then see that (5.19), with the help of Lemma 4.2, becomes

$$z^k M_1(z)\, q' = z^{k-1} R(z)\, q + z^{k-1} f_2(z). \tag{5.23}$$

The matrices $M_1(z)$ and $R(z)$ are those appearing by the same names in (5.7) and (5.12). The vector $f_2(z)$ is holomorphic at $z = 0$. Notice that since the constant terms of the matrix $M_1(z)$ are an upper triangular matrix, we rewrite (5.23) in the form

$$zq' = H(z)\, q + \psi_2(z), \tag{5.24}$$

where

$$\psi_2(z) = M_1^{-1}(z)\, f_2(z)$$

also has the property that its last component has a zero at $z = 0$. This last mentioned fact allows us to find a formal power series solution for (5.24). That this formal series actually converges follows from Lemma 7.1 of [2].

The remainder of the proof of Theorem 5.1 consists in carrying out an argument by induction to show that we may solve the remaining Eqs. (2.5) for $\mu > 2$. We shall only mention here that in so doing, one again chooses $B_\mu(z)$ in the same manner as $B_2(z)$. This leads to a system having the same form as (5.24).

The proof of Theorem 5.1 is then completed upon a formal recording of the series for $B_1(z, \epsilon)$.

6. Reduction Theorems for Turning Point Problems of Type $T_n(1, k, n-1)$

In this section we investigate the case where

$$a_n{}^*(z) = z^{k+1} a_n(z)$$

$$a_j^*(z) = z^k a_j(z), \qquad (j = 1,\dots, n-1),$$

with the functions $a_j(z)$, $(j = 1,\dots, n)$, holomorphic at $z = 0$ and $a_{n-1}(0) \neq 0$.
We will prove the following theorem.

Theorem 6.1. *There exists a formal transformation $y = P(z, \epsilon)\, y^*$ which reduces problems of type $T_n(1, k, n-1)$ to the form*

$$\epsilon y^{*\prime} = B(z, \epsilon)\, y^*,$$

where

$$B(z, \epsilon) = A_0(z) + \epsilon \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & & & & & 0 \\ 0 & & & & & \vdots \\ \vdots & & & & & 0 \\ \sum_{m=0}^{2k-2} b_{1m} z^m, & & \sum_{m=0}^{k-2} b_{2m} z^m, & \cdots, & & \sum_{m=0}^{k-2} b_{nm} z^m \end{bmatrix}$$

$$= A_0(z) + \epsilon B_1(z, \epsilon).$$

The b_{jm} are formal power series in ϵ with constant coefficients.

We now begin the proof of Theorem 6.1.

As in Section 5, Eq. (2.3) is solved by a matrix $P_0(z)$ as given in (5.1).

The conditions of compatibility for (2.4) are given in (5.2). This leads to the system (5.3) where now

$$S(z) = \{\operatorname{tr} A_0^{n-k+r}\}, \qquad (k = 1,\dots, n)$$
$$(r = 0,\dots, n-1), \tag{6.1}$$

while

$$T(z) = \{\operatorname{tr} [A_0^{n-k+r}(A_1 - B_1) - (n-k)\, A_0^{n-k+r-1} A_0{}']\}$$
$$= T_1(z) - T_2(z), \tag{6.2}$$

where the matrices T_1 and T_2 are self-explanatory.

The integer k denotes the columns while $r + 1$ denotes the rows of the matrices appearing in (6.1) and (6.2).

From Lemmas 4.3, 4.4 and 4.5 we see that

$$S(z) = z^k \left[\frac{n}{z^k} K_1 + K_2 + (n-1)\, z^k a_{n-1}^2(z)\, K_3 + 0(z^{k+1})\right]. \tag{6.3}$$

Here the matrix K_1 is zero except for a single entry of value one in the first row and last column. K_3 is a matrix of functions holomorphic at $z = 0$ whose entry in the last row and first column is identically one. The matrix K_2 has the form

$$K_2 = \begin{bmatrix} (n-1)\, a_{n-1}, & (n-2)\, a_{n-2}, & \cdots, & a_1, & 0 \\ 0, & (n-1)\, a_{n-1}, & \cdots, & a_2, & a_1 \\ \cdot & \cdot & & & \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ 0, & & \cdots, & 0, & (n-1)\, a_{n-1} \end{bmatrix}. \tag{6.4}$$

Let us put

$$M_1(z) = \frac{S(z)}{z^k}, \qquad (z \neq 0).$$

We wish to calculate

$$M_1^{-1}(z) = \left(\frac{n}{z^k} K_1 + K_2 + (n-1)\, a_{n-1}^2 z^k K_3 + 0(z^{k+1})\right)^{-1}.$$

Now

$$\left(\frac{n}{z^k} K_1 + K_2\right)^{-1} = K_2^{-1} - \frac{n}{z^k[(n-1)\, a_{n-1}]^2} K_1, \tag{6.5}$$

so that

$$M_1^{-1}(z) = \left[\left(\frac{n}{z^k} K_1 + K_2\right) K_4(I + 0(z))\right]^{-1}, \tag{6.6}$$

where

$$K_4 = I - \frac{nK_1K_3}{n-1} = \begin{bmatrix} -\dfrac{1}{n+1}, & * & *, & \cdots & * \\ 0 & 1 & 0 & \cdots & 0 \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ \cdot & & & & \cdot \\ 0 & & \cdots & 0 & 1 \end{bmatrix}. \tag{6.7}$$

From Lemmas 4.2 and 4.3 we see that with $a_{1j} = \operatorname{tr}(A_1 A_0^{n-j})$, $(j = 1, \ldots, n)$, we will have

$$T_1(z) = \begin{bmatrix} a_{11} - b_{11}, & a_{12} - b_{12}, & \cdots, & a_{1n} - b_{1n} \\ 0, & a_{11} - b_{11}, & \cdots & \cdot \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ \cdot & & & \cdot \\ 0, & \cdots & 0, & a_{11} - b_{11} \end{bmatrix}$$

$$+ z^k \cdot \begin{bmatrix} & 0(1) & \\ \sum 0(a_{1j} - b_{1j}), & \cdots, & \sum 0(a_{1j} - b_{1j}) \end{bmatrix} + 0(z^{2k}), \tag{6.8}$$

and

$$\begin{aligned} T_2(z) &= \{(n-k) \operatorname{tr}(A_0^{n-k+r-1} A_0')\} \\ &= \left\{ \frac{n-k}{n-k+r} \operatorname{tr}(A_0^{n-k+r})' \right\} \\ &= \left\{ \frac{n-k}{n-k+r} \operatorname{tr}(A_0^{n-k+r}) \right\}'. \end{aligned} \tag{6.9}$$

The expressions $\sum 0(a_{1j} - b_{1j})$ which occur in (6.8) designate sums of functions holomorphic at $z = 0$ all of which are multiplied by the (as yet unspecified) holomorphic functions $a_{1j} - b_{1j}$, $(j = 1, \ldots, n)$.

We may then write the system (5.3) in the form

$$\begin{aligned} z^k q' &= M_1^{-1}(z)\, T(z)\, q \\ &= R(z)\, q, \end{aligned} \tag{6.10}$$

where

$$R(z) = [I + 0(z)]\, K_4^{-1} \left[K_2^{-1} - \frac{n}{z^k[(n-1)\, a_{n-1}]^2} K_1 \right] [T_1(z) - T_2(z)]. \tag{6.11}$$

Now

$$K_1(T_1 - T_2)(z) = z^k \begin{bmatrix} \sum 0(a_{1j} - b_{1j}), & \cdots, & \sum 0(a_{1j} - b_{1j}) \\ & 0 & \end{bmatrix}$$

$$+ \begin{bmatrix} 0, & \cdots, & 0, & a_{11} - b_{11} \\ \cdot & \cdot & & 0 \\ \cdot & & & \cdot \\ \cdot & & \cdot & \cdot \\ & & \cdot & \cdot \\ 0, & \cdots, & 0, & 0 \end{bmatrix} + 0(z^{2k-1}). \tag{6.12}$$

Since the last row of $T_2(z)$ is $0(z^{2k-1})$, we see from (6.11) and (6.12) that $R(z)$ will be both holomorphic at $z = 0$ and divisible by z^{k-1} if and only if $a_{11} - b_{11} = 0\,(z^{2k-1})$ and

$$a_{1j} - b_{1j} = 0\,(z^{k-1}), \qquad (j = 2,\ldots, n).$$

Suppose, then, that we expand the functions $\{a_{1j}(z)\}$ in their power series at $z = 0$.

Then

$$a_{1j}(z) = \sum_{p=0}^{\infty} a_{1jp} z^p, \qquad (j = 1,\ldots, n).$$

Let

$$b_{11}(z) = \sum_{p=0}^{2k-2} a_{11p} z^p,$$

and

$$b_{1j}(z) = \sum_{p=0}^{k-2} a_{1jp} z^p, \qquad (j = 2,\ldots, n).$$

With this choice,

$$R(z) = z^{k-1} H(z),$$

where $H(z)$ is holomorphic at $z = 0$ and

$$H(0) = \begin{bmatrix} -k & & & & & \\ 0 & -\dfrac{(n-2)\,k}{n-1} & & & * & \\ 0 & & -\dfrac{(n-3)\,k}{n-1} & & & \\ \cdot & & & \cdot & & \\ \cdot & & & & \cdot & \\ \cdot & & & & & \cdot \\ & & & & \dfrac{-k}{n-1} & \\ 0, & & \cdots & & 0, & 0 \end{bmatrix}. \tag{6.13}$$

The system (6.10) then has the form

$$zq' = H(z)\, q, \tag{6.14}$$

where $H(z)$ is holomorphic at $z = 0$. From (6.13) we see that no eigenvalue of $H(0)$ is a positive integer while one eigenvalue is zero. Hence we may find a formal power series solution for (6.14) with $q_n(0) = 1$. By a classical theorem of systems of ordinary differential equations with a regular singular point, this formal series solution will converge for $|z| < \delta_0$.

We have now determined P_0 as a nonsingular matrix holomorphic at $z = 0$ such that the conditions of compatibility for the equation (2.4) are satisfied. By Lemma 2.3, Eq. (2.4) has a particular holomorphic solution, say $\tilde{P}_1(z)$.

The most general solution of (2.4) is of the form

$$P_1(z) = \tilde{P}_1(z) + \mathring{P}_1(z), \tag{6.15}$$

where

$$\mathring{P}_1(z) = \sum_{k=1}^{n} q_k(z)\, A_0^{n-k}(z). \tag{6.16}$$

The scalar functions $q_k(z)$, $(k = 1,..., n)$, are arbitrary and are not, in general, the same as those appearing in the first part of Section 6.

In order to solve (2.5) with $\mu = 2$, it is necessary and sufficient that

$$\operatorname{tr}\,[P_1' - (A_1P_1 - P_1B_1) + P_0B_2 - F_2)\, A_0{}^k] = 0, \qquad (k = 0, 1,..., n-1). \tag{6.17}$$

(6.17) then becomes

$$S(z)\, q' = T(z)\, q + \phi_2(z), \tag{6.18}$$

where $\phi_2(z)$ is given in (5.20). Let the functions $\mathring{f}_{2j}$, $(j = 1,..., n)$, have the same definitions as in Section 5, and suppose that they have been expanded in their power series at $z = 0$ as in (5.21). We now define the matrix $B_2(z)$.

Make every entry zero except in the last row. Put

$$b_{21}(z) = \sum_{\nu=0}^{2k-2} \mathring{f}_{21\nu} z^\nu$$

and

$$b_{2j}(z) = \sum_{\nu=0}^{k-2} \mathring{f}_{2j\nu} z^\nu, \qquad (j = 2,..., n). \tag{6.19}$$

We then see that (6.18) becomes

$$z^k q' = z^{k-1} H(z)\, q + f_2(z), \tag{6.20}$$

where

$$f_2(z) = [I + 0(z)]\, K_4^{-1} \left(K_2^{-1} - \frac{n}{z^k[(n-1)\, a_{n-1}]^2}\, K_1\right) \phi_2(z).$$

From Lemma 4.2 and (6.19) we see that the last component of $\phi_2(z)$ is $0(z^{2k-1})$, while the remaining components are $0(z^{k-1})$. Hence, $f_2(z)$ is both holomorphic at $z = 0$ and divisible by z^{k-1}, so that (6.20) has the form

$$zq' = H(z)\, q + \psi_2(z), \tag{6.21}$$

where the vector $\psi_2(z)$ is holomorphic at $z = 0$.

Also, the last component of $\psi_2(z)$ has a zero at $z = 0$. This can be seen by noting that the constant terms of $K_4^{-1}K_2^{-1}$ form an upper triangular matrix, and that $\phi_2(z)$ is $0(z^{2k-1})$ in the last component.

Thanks to this last mentioned fact, (6.21) has a formal power series solution. That this series actually converges is proved in Lemma 7.1 of [2].

Apart from a straightforward induction proof, we may solve the equations (2.5) for $\mu > 2$. Theorem 6.1 is proved upon formally reordering the series for $B_1(z, \epsilon)$.

7. The Analytic Meaning of the Formal Reduction

It should be observed that in Sections 5 and 6 we have been careful to mention that the simplification of the differential Equation (1.1) was only formal.

The results of these two sections together with a theorem of J. Ritt can be used to give the following analytic result.

Let $B(z, \epsilon)$ and $P(z, \epsilon)$ be matrices which have the formal series $\Sigma B_r(z)\, \epsilon^r$ and $\Sigma P_r(z)\, \epsilon^r$ obtained in Sections 5 and 6 as their asymptotic expansions in the region D described in (1.2). That such matrices exist is the essence of Ritt's theorem. The matrix $P(z, \epsilon)$ is nonsingular in D.

From our formal simplifications of (1.1) made in Sections 5 and 6, the transformation $y = P(z, \epsilon)w$ reduces (1.1) with $h = 1$ to

$$\epsilon w' = [B(z, \epsilon) + \Omega(z, \epsilon)]w,$$

where $\Omega(z, \epsilon) \sim 0$ as $\epsilon \to 0$, $(\epsilon \in D)$.

That $\Omega(z, \epsilon)$ is identically zero can be concluded in the important special case where the functions appearing in the leading matrix $A_0(z)$ are polynomials in z, provided certain additional restrictive hypotheses are satisfied. The precise statement, which we omit here because of its length, follows directly from [4].

The analytic portion of the formal reductions in the general case where $A_0(z)$ is not a polynomial appears to be a rather formidable open problem that deserves further study.

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